Utility for Money and Consumer Budgets

It is often convenient to narrow the choice of utilities further and express utility in monetary terms. To do so, one can pose the question: Given the customer's preference for n goods (purchase alternatives), a vector of market prices $\mathbf{p} = (p_1, \ldots, p_n)$ for these goods, and a level of monetary wealth w, how would a customer "spend" his wealth? To make matters simpler, we assume quantities x_i of each good i are continuous and our customer has a continuous utility function $u(\mathbf{x})$. Let $\mathbf{x} = (x_1, \ldots, x_n)$. The *consumer budget problem* can then be formulated as[1]

$$\begin{aligned} v(w) = \quad & \max \quad u(\mathbf{x}) \\ & \text{s.t.} \quad \mathbf{p}^\top \mathbf{x} \leq w \\ & \qquad\quad \mathbf{x} \geq 0. \end{aligned} \tag{E.1}$$

In other words, customers purchase quantities x_i of each good i to maximize their total utility subject to the constraint that they can spend at most their total wealth w. The optimal solution gives the customer's utility for wealth (or money) $v(w)$; the optimal solution, $\mathbf{x}^*$, gives the customer's *demand* for each of the n goods.

Utility for money is increasing in w since one can always "not spend" the wealth w. Also, since the utility for money depends on the prices of goods, if prices change, both the demand x^* and the utility for money may change. The *marginal utility of money* $u'(w)$ also depends on the customer's wealth w. The utility for money $v(w)$ is concave if $u(x)$ is concave,[2] in which case the consumer has decreasing marginal utility for money. Intuitively, this is because at low levels of wealth only highly essential goods are purchased (food, water, clothing, shelter)—all of which have very high utility to most of us. As wealth rises, each marginal dollar is allocated to somewhat less important purchases.

If the function $u(\mathbf{x})$ is continuously differentiable and we let π denote the optimal Lagrange multiplier on the budget constraint in (E.1), then the marginal value of money is

$$v'(w) = \pi.$$

We can use this fact to redefine utilities in monetary terms. Indeed, since our customer's monetary utility for an additional dollar should be one dollar, we should have $v'(w) = \pi = 1$ if utilities are measured in dollars. This change of units can be accomplished by rescaling the customer's utility functions by $v'(w) = \pi$ to form the modified utilities

$$\tilde{u}(\mathbf{x}) = \frac{u(\mathbf{x})}{\pi}. \tag{E.2}$$

[1]Dynamic versions of this consumer budget problem can also be formulated by allowing customers to purchase over multiple periods and invest money at a given interest rate for future consumption. Other variations introduce wages and a utility for leisure time and allow customers to increase their monetary wealth by varying their time allocated to labor, and so on.

[2]This follows easily from the convexity of the budget constraint and the fact that (E.1) is a maximization problem. Concavity of the utility function corresponds to having decreasing marginal utility of consumption for goods, which is a natural assumption.